THE REAL COHOMOLOGY RING OF A SPHERE BUNDLE OVER A DIFFERENTIABLE MANIFOLD

BY

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1. **Introduction.** The main result of this paper is already known; it follows from a theorem of A. Borel [2, paragraph 24] and has also been obtained by G. Hirsch [3]. However, it is believed that the proofs and descriptions of results given here are much simpler than in either of the above papers. The method used here is that of a previous paper [4] of the author; this paper could serve as an introduction to it.

We are concerned with the cohomology ring with real coefficients of a fiber space whose fiber is a $(k-1)$-sphere, for k even. When given a cohomology theory whose cochains are anti-commutative, a description is given of the cohomology ring of the total space, in terms of the characteristic class W_k of the sphere space, which is practical for computations in many cases. In §5 these results are used to obtain, in a simplified form, the results of G. Hirsch [3] in terms of his "strict" triple products.

If the fibering sphere is of *even* dimension, then using real coefficients, $W_k=0$, and the cohomology ring is described in the paper [5] of W. S. Massey.

The author wishes to thank W. S. Massey for pointing out the simplification which occurs when given an anti-commutative ring of cochains, and for help in the preparation of this paper.

2. We suppose given a fiber space (E, p, B, S^{k-1}) with fiber a $(k-1)$-sphere, k even. By "fiber space" is meant a "locally trivial fiber space": For each $x\in B$, there is a neighborhood V of x and a homeomorphism ϕ mapping $V\times S^{k-1}$ onto $p^{-1}(V)$ such that $p\phi(y, z)=y$ for $y\in V$ and $z\in S^{k-1}$. We will assume that the sphere space is orientable in the following sense: If S_x^{k-1} denotes the fiber over $x\in B$, then the local system of groups defined by $H^{k-1}(S_x^{k-1})$, for $x\in B$, is a simple system. We also assume that the base space B is compact, and that we are given a cohomology theory with real coefficients, whose cochains are anti-commutative. An example of this would be if B were a differentiable manifold, the sphere space had a differentiable structure, and the ring of cochains were the exterior differential forms. However, it is possible to have a ring of anti-commutative cochains under less restrictive assumptions.

We recall a result of Thom [7]: If A is the mapping cylinder of $p: E\rightarrow B$, the Gysin sequence of (E, p, B, S^{k-1}) is isomorphic to the cohomology sequence of the pair (A, E). In fact, there is an element $\mathfrak{U}\in H^k(A, E)$ such that

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the homomorphism $\theta: H^{q-k}(A) \to H^q(A, E)$ defined by $\theta(x) = x\mathfrak{U}$ (the cup product) is an isomorphism onto. The natural projection $p_0: A \to B$ of the mapping cylinder onto the base space induces an isomorphism $p_0^*: H^q(B) \to H^q(A)$. We then have the commutative diagram

$$\begin{array}{ccccccc}
\cdots \to H^{q-k}(B) & \xrightarrow{\mu} & H^q(B) & \xrightarrow{p^*} & H^q(E) & \xrightarrow{\psi} & H^{q-k+1}(B) \to \cdots \\
\downarrow p_0^* & & \Big| & & \Big| & & \downarrow p_0^* \\
H^{q-k}(A) & & \Big| p_0^* & & \Big| \text{id.} & & H^{q-k+1}(A) \\
\downarrow \theta & & \downarrow & & \downarrow & & \downarrow \theta \\
\cdots \to H^q(A, E) & \xrightarrow{n^*} & H^q(A) & \xrightarrow{m^*} & H^q(E) & \xrightarrow{\delta^*} & H^{q+1}(A, E) \to \cdots
\end{array}$$

FIGURE 1

In this diagram, the top line is the Gysin sequence and the bottom line is the cohomology sequence of the pair (A, E). According to the results of Thom, $n^*(\mathfrak{U}) = p_0^*(W_k)$.

We will regard $C^*(A, E)$ as a subgroup of $C^*(A)$. It is actually an ideal in $C^*(A)$ and $C^*(E) \approx C^*(A)/C^*(A, E)$. Making this identification, n^* is induced by the inclusion $C^*(A, E) \subset C^*(A)$, and m^* is induced by the natural map $m^{\#}: C^*(A) \to C^*(A)/C^*(A, E)$, where $m: E \to A$ is the inclusion.

3. Let V be a representative cocycle for the characteristic class $W_k \in H^k(B)$. Construct the algebraic mapping cylinder M of the map $x \to xV$ for $x \in C^*(B)$, that is, let

$$M^p = C^p(B) \times C^{p-k+1}(B),$$
$$M = \sum_p M^p,$$

and

$$\delta(x, y) = (\delta x + yV, -\delta y) \qquad \text{for } (x, y) \in M.$$

It is easily seen that (M, δ) is a differential group, with addition defined componentwise.

We now introduce a multiplication in M, by the formula

$$(x, y)(v, w) = (xv, (-1)^p xw + yv)$$

for $(x, y) \in M^p$ and $(v, w) \in M^q$. The product has degree $p+q$, and is associative and anti-commutative. It is easily verified that $\delta[(x, y)(v, w)] = [\delta(x, y)](v, w) + (-1)^p(x, y)\delta(v, w)$ for $(x, y) \in M^p$. Consequently a product is induced in the derived group $H^*(M)$.

THEOREM I. *Using this product, $H^*(M)$ is isomorphic to $H^*(E)$ as an algebra over the reals.*

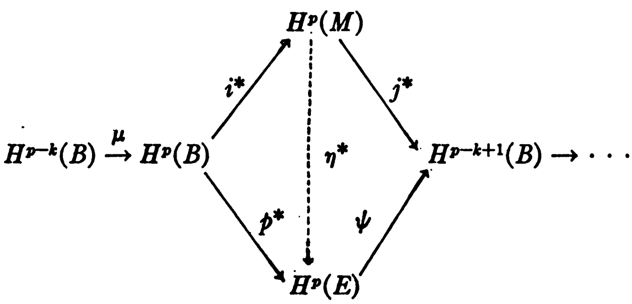
To prove this theorem, we consider the diagram of exact sequences

$$\begin{array}{ccccccccc} 0 \to & C^p(B) & \xrightarrow{i} & M^p & \xrightarrow{j} & C^{p-k+1}(B) & \to 0 \\ & \downarrow \delta & & \delta \downarrow & & \downarrow \delta & \\ 0 \to & C^{p+1}(B) & \xrightarrow{i} & M^{p+1} & \xrightarrow{j} & C^{p-k+2}(B) & \to 0 \end{array}$$

where $i(x)=(x, 0)$ for $x \in C^*(B)$ and $j(x, y)=y$ for $(x, y) \in M$. The left square commutes and the right square anti-commutes. From this diagram we obtain the exact cohomology sequence of the algebraic mapping cylinder:

$$\cdots \to H^{p-k}(B) \xrightarrow{\mu} H^p(B) \xrightarrow{i^*} H^p(M) \xrightarrow{j^*} H^{p-k+1}(B) \to \cdots .$$

Here, μ is induced by $x \to xV$, in other words μ is the same map as in the Gysin sequence. We have the diagram



We will define a ring homomorphism $\eta^*: H^p(M) \to H^p(E)$ for which $\eta^* i^* = p^*$ and $-j^* = \psi\eta^*$; it will then follow by the five-lemma that η^* is an isomorphism onto.

To define η^*, let $V' \in C^k(A, E)$ be a representative cocycle for Thom's class $\mathfrak{U} \in H^k(A, E)$. Then for some $\alpha \in C^{k-1}(A)$, $p_0^\#(V) = V' + \delta\alpha$. Here, $p_0^\#$ is the cochain homomorphism induced by $p_0: A \to B$. Define

$$\eta: M^p \to C^p(A)/C^p(A, E) = C^p(E)$$

by

$$\eta(x, y) = m^\#(p_0^\#(x) + (-1)^{p+1}(p_0^\# y)\alpha) \qquad \text{for } (x, y) \in M^p.$$

It is easily verified that $\delta\eta = \eta\delta$ and that η is an additive homomorphism. Also, for $(x, y) \in M^p$ and $(u, v) \in M^q$, a straightforward computation shows that

$$\eta\{(x, y)(u, v)\} - \eta(x, y)\eta(u, v) = m^\#(\Gamma)$$

where

$$\Gamma = (-1)^{p+q+1} p_0^\sharp(yu)\alpha + (-1)^p (p_0^\sharp y)\alpha(p_0^\sharp u) + (-1)^{p+q+1}(p_0^\sharp y)\alpha(p_0^\sharp v)\alpha.$$

$\Gamma = 0$ since the cochains are anti-commutative, and $\alpha^2 = 0$; therefore the induced map $\eta^*: H^*(M) \rightarrow H^*(E)$ is a ring homomorphism.

To prove the commutativity relations, let square brackets denote cohomology classes in the appropriate cohomology groups. For $[x] \in H^p(B)$, $\eta^* i^*[x] = \eta^*[(x, 0)] = [m^\sharp p_0^\sharp x] = p^*[x]$, in view of the commutativity of Figure 1. For $[(x, y)] \in H^p(M)$,

$$\begin{aligned}(\psi\eta^*)[(x, y)] &= \psi[m^\sharp(p_0^\sharp x + (-1)^{p+1}(p_0^\sharp y)\alpha)] \\ &= p_0^{*-1}\theta^{-1}\delta^*[m^\sharp(p_0^\sharp x + (-1)^{p+1}(p_0^\sharp y)\alpha)] \\ &= p_0^{*-1}\theta^{-1}[p_0^\sharp(\delta x) + (p_0^\sharp y)(p_0^\sharp V - V')] \\ &= p_0^{*-1}\theta^{-1}[-(p_0^\sharp y)V'] = -[y] = -j^*[(x, y)].\end{aligned}$$

We remark that the algebra $H^*(M)$ is independent of the choice $V \in W_k$, as this theorem shows.

4. In this section we prove two propositions which should be useful in actual computations. Suppose C and $\overline{C}$ are graded, anti-commutative cochain rings, and V and $\overline{V}$ are C and $\overline{C}$ k-cocycles, respectively. Let $f: C \rightarrow \overline{C}$ be an allowable homomorphism for which $f(V) = \overline{V}$. Let M and $\overline{M}$ be the algebraic mapping cylinders of $x \rightarrow xV$ and $x \rightarrow x\overline{V}$ respectively.

Proposition A. *There is an allowable homomorphism $\phi: M \rightarrow \overline{M}$, which preserves products, induced by f. The homomorphism $\phi^*: H^*(M) \rightarrow H^*(\overline{M})$ induced by ϕ commutes with the homomorphisms of the algebraic mapping cylinders. Furthermore if $f^*: H^*(C) \rightarrow H^*(\overline{C})$ is an isomorphism onto, then so is ϕ^*.*

To prove this, define ϕ by $\phi(x, y) = (fx, fy)$ for $(x, y) \in M$. All the computations are straightforward. The last assertion follows from the five-lemma.

Now let M^n be a compact, connected, differentiable n-dimensional manifold, with $H^1(M^n) = 0$. Let C^* be the algebra of exterior differential forms, with differential operator d.

Proposition B[1]. *It is possible to choose a finitely generated subring A^* of C^*, for which the inclusion $i: A^* \rightarrow C^*$ induces an isomorphism onto $i^*: H^*(A^*) \rightarrow H^*(C^*)$.*

Proof. Let $1, a_1, \cdots, a_k$ be a minimal set of generators of $H^*(C^*)$, with representative cocycles $1', a_1', \cdots, a_k'$. These generate $A \subset C^*$, and the inclusion $i: A \rightarrow C^*$ induces $i^*: H^*(A) \rightarrow H^*(C^*)$ which is onto. The kernel of i^* is generated as an ideal by $b_1, \cdots, b_l$; choose $b_1', \cdots, b_l' \in C^*$ for which $db_i' = b_i$. Let A' be generated by $1, a_1', \cdots, a_k', b_1', \cdots, b_l'$; make this set of

(1) This was suggested by W. S. Massey.

generators minimal. All of the b_i have degree at least three, for they arise from cocycles a_i', a_j' for which $a_i' a_j'$ is a coboundary, and $a_i' a_j'$ has degree at least four. If $i^{*\prime}: H^*(A') \to H^*(C^*)$ is not 1-1, repeat the process, adding c_i's which have degree at least four. The process must end, for $C^p = 0$ for $p > n$.

Using these two propositions, one can compute the cohomology rings in many cases.

5. We now suppose that the base space B is a differentiable manifold, with a Riemannian metric, and we will use the exterior differential forms as cochains on B. Under these conditions, there are canonical additive homomorphisms (Z and $\mathcal{B}$ represent cocycles and coboundaries)

$$\alpha: H^p(B) \to Z^p(B)$$

and

$$\beta: \mathcal{B}^p(B) \to C^{p-1}(B).$$

α assigns to each cohomology class a representative cocycle (a harmonic form) and β assigns to each p-coboundary a $(p-1)$-cochain such that $\delta\beta$ = identity. (See [1] or [6]; an outline of the results is given in [3].)

We construct the algebraic mapping cylinder of $x \to x\alpha(W_k)$ for $x \in C^*(B)$. This gives the exact sequence

$$0 \to \frac{H^p(B)}{W_k \cdot H^{p-k}(B)} \xrightarrow{i} H^p(M) \xrightarrow{j} (\text{kernel } \mu)^{p-k+1} \to 0$$

where i and j are induced by i^* and j^* in the obvious ways, and

$$(\text{kernel } \mu)^{p-k+1} = (\text{kernel } \mu) \cap H^{p-k+1}(B).$$

Note that a multiplication

$$\frac{H^p(B)}{W_k \cdot H^{p-k}(B)} \times \text{kernel } \mu \to \text{kernel } \mu$$

is defined. This sequence splits; we define θ: $(\text{kernel } \mu)^{p-k+1} \to H^p(M)$ by requiring $\theta(y)$ to be the cohomology class in $H^*(M)$ of $(-\beta(\alpha y \cdot \alpha W_k), \alpha y)$. It is easily verified that θ is additive and $j\theta$ = identity. Thus every $x \in H^*(M)$ may be written uniquely in the form $x = i(x_1) + \theta(x_2)$, where $x_2 = j(x)$. The product of two elements may be computed as follows (p is the degree of x_1):

$$(i(x_1) + \theta(x_2))(i(y_1) + \theta(y_2)) = i(x_1y_1) + \theta(x_2)i(y_1) + i(x_1)\theta(y_2) + \theta(x_2)\theta(y_2).$$

Applying j to both sides, we get

$$(*) \qquad \begin{aligned} &j\{(i(x_1) + \theta(x_2))(i(y_1) + \theta(y_2))\} \\ &\quad = j(\theta(x_2)i(y_1)) + j(i(x_1)\theta(y_2)) + j(\theta(x_2)\theta(y_2)) \\ &\quad = x_2y_1 + (-1)^p x_1y_2 + j(\theta(x_2)\theta(y_2)), \end{aligned}$$

since $j(\theta(x_2)i(y_1)) = x_2y_1$ and $j(i(x_1)\theta(y_2)) = (-1)^p x_1y_2$. To prove this last relation, let $x_1' \in H^p(B)$ represent x_1; then $i(x_1)$ is represented by $(\alpha x_1', 0) \in M^p$. $\theta(y_2)$ is represented by $(-\beta[(\alpha y_2)(\alpha W_k)], \alpha y_2)$. Using the product formula in M^p, we obtain $j(i(x_1)\theta(y_2)) = (-1)^p x_1 y_2$. The other formula is proved similarly.

From the above equation (*) we obtain

$$(i(x_1) + \theta(x_2))(i(y_1) + \theta(y_2)) = i(x_1 y_1) + \theta(x_2 y_1 + (-1)^p x_1 y_2 + j(\theta x_2 \cdot \theta y_2)).$$

We may compute $j(\theta x_2 \cdot \theta y_2)$; it is the cohomology class of

$$-(\alpha x_2)\beta(\alpha y_2 \cdot \alpha W_k) + (-1)^{p+1}\beta(\alpha x_2 \cdot \alpha W_k)\alpha(y_2).$$

However, this is just $(-1)^{p+1}\langle x_2, W_k, y_2\rangle$, the triple product *in the strict sense* of Hirsch [3]. We thus have the following theorem.

THEOREM II. *For the base space B a differentiable manifold with Riemannian metric, we have*

$$H^p(E) \approx \frac{H^p(B)}{W_k \cdot H^{p-k}(B)} \oplus (\text{kernel } \mu)^{p-k+1}.$$

The multiplication is given in terms of the strict triple product by

$$(x_1, x_2)(y_1, y_2) = (x_1 y_1, x_2 y_1 + (-1)^p x_1 y_2 + (-1)^{p+1}\langle x_2, W_k, y_2\rangle).$$

Theorem II is simpler than the corresponding result of Hirsch [3]; Hirsch uses several cohomology operations. In the above formulation, x_1 and y_1 are elements of $H^*(B)/W_k \cdot H^*(B)$. The author believes that the extra operations that Hirsch uses only lead to different representative elements for x_1 and y_1, and thus give the same product.

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